

Bayesian Games, Social Welfare Solutions and Quantum Entanglement

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Entanglement is of paramount importance in quantum information theory. Its supremacy over classical correlations has been demonstrated in a numerous information theoretic protocols. Here we study possible adequacy of quantum entanglement in Bayesian game theory, particularly in social welfare solution (SWS), a strategy which the players follow to maximize sum of their payoffs. Given a multi-partite quantum state as an advice, players can come up with several correlated strategies by performing local measurements on their parts of the quantum state. A quantum strategy is called quantum-SWS if it is advantageous over a classical equilibrium (CE) strategy in the sense that none of the players has to sacrifice their CE-payoff rather some have incentive and at the same time it maximizes sum of all players' payoffs over all possible quantum advantageous strategies. Quantum state yielding such a quantum-SWS is coined as quantum social welfare advice (SWA). Interestingly, we show that any two-qubit pure entangled states, even if it is arbitrarily close to a product state, can serve as quantum-SWA in some Bayesian game. Our result, thus, gives cognizance to the fact that every two-qubit pure entanglement is the best resource for some operational task.

Introduction.— Game theory is the study of human conflict and cooperation within a competitive situation. It has been widely used in various social and behavioral sciences, *e.g.*, economics [1], political sciences [2], biological phenomena [3], as well as logic, computer science, and psychology [4]. More formally, game theory is a mathematical study of strategic decision making among interacting decision makers. Each decision maker is considered as a player with a set of possible actions and each one has preference over certain actions. Mathematically such preference can be modeled by associating some payoff with each of the action. First systematic study of preferences over different possible actions was discussed by von Neumann and Morgenstern [5]. Then J. Nash introduced the seminal concept— the concept of *Nash equilibrium* [6]. He also proved that for any game, with finite number of actions for each player, there will always be a mixed strategy Nash equilibrium. Later, Harsanyi introduced the notion of Bayesian games or games of incomplete information where each player have some private information unknown to other players [7]. In such a Bayesian scenario Aumann proved that the proper notion of equilibrium is not the ordinary mixed strategy Nash equilibrium but a more general – *correlated equilibrium* [8]. A correlated equilibrium can be achieved by some correlated strategy with correlation given to the players as common advice by some referee. Psychology of the players participating in a game is also an important component in the study of game theory [9].

Psychological evidence shows that rather than pursuing solely their own payoffs, players may also consider additional social goals. Such social behavior of the players may result different types of 'fairness equilibrium' solution. One such concept is *social welfare solution* (SWS) where the players try to maximize sum of their payoffs [10].

In this work, we study this particular notion of SWS, but in the quantum realm. In the quantum scenario the referee, instead of a classical correlation, provides a multi-partite quantum state to the players as common advice. The players can come up with correlations generated from the quantum advice by performing local measurements on their respective parts of quantum state and consequently follow a correlated strategy. Such a quantum strategy is advantageous over a classical equilibrium (CE) strategy if no one's payoff gets decreased from the CE-payoff rather some players have incentive over the CE-payoff while following this quantum strategy. Among different quantum advantageous strategies those maximizing the sum of all players' payoffs is called quantum-SWS. Quantum states producing such strategy are called quantum social welfare advices (quantum-SWA). In this work we show that any two-qubit pure entangled state, however small entanglement it may have, can produce quantum-SWS for some Bayesian game. In other words, all such entangled states can act as the best useful resource for some tasks. We establish this claim by constructing a family of two-player Bayesian games. But in the following first we briefly discuss preliminaries of game theory.

Preliminaries of game theory.— The very basic assumption made in game theory is that the players are rational,

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i.e., they will choose the best actions to get highest available payoffs [11]. We denote a game by the symbol G and for simplicity we restrict the discussion in two-player games played between (say) Alice and Bob (extension to higher number of players is straightforward and interested readers may see the classic book of Osborne [4]). We denote the type of i^{th} player by $t_i \in \mathcal{T}_i$ and denote her/his action by $s_i \in \mathcal{S}_i$, for $i \in \{A, B\}$, calligraphic fonts denoting the type and action profiles. A type can represent many things: it can be a characteristic of the player or a secret objective of the player, which remain private to the players in Bayesian scenario. There may be a prior probability distribution $P(\mathcal{T})$ over the type profile $\mathcal{T} := \mathcal{T}_A \times \mathcal{T}_B$. Each player is given a payoff over the type and action profile, i.e., $v_i : \mathcal{T} \times \mathcal{S} \mapsto \mathbb{R}$, where $\mathcal{S} := \mathcal{S}_A \times \mathcal{S}_B$. In the absence of any correlation or external advice, players can apply strategies that are either pure or mixed. For the i^{th} player, a pure strategy is a map $g_i : \mathcal{T}_i \mapsto \mathcal{S}_i$, meaning that the player select a deterministic action based only on her/his type. A mixed strategy is a probability distribution over pure ones, i.e. the function $g_i : \mathcal{T}_i \mapsto \mathcal{S}_i$ becomes a random function described by a conditional probability distribution on \mathcal{S}_i given the type $t_i \in \mathcal{T}_i$ and we will denote such mixed strategies as $g_i(s_i|t_i)$ (for a more detailed discussion see [12]). The average payoff for the i^{th} player is given by, $\langle v_i(g) \rangle \equiv \mathbb{E}_{t,g} v_i(t, g_A(t_A), g_B(t_B)) := \sum_{t,s} P(t) v_i(t, s) g_A(s_A|t_A) g_B(s_B|t_B)$. Here $g \equiv (g_A, g_B) \in \mathcal{G} = \mathcal{G}_A \times \mathcal{G}_B$, with \mathcal{G}_i denoting the strategy profile for the i^{th} party, $s \equiv (s_A, s_B) \in \mathcal{S}$, and $t \equiv (t_A, t_B) \in \mathcal{T}$; and $P(t)$ denote the probability according to which the types are sampled. A solution for a game is a family of strategies $g \equiv (g_A, g_B)$, each for Alice and Bob respectively. A solution g^* is a Nash equilibrium (uncorrelated) if no player has an incentive to change the adopted strategy, i.e., $\langle v_i(g^*) \rangle \geq \langle v_i(g_i, g_{-i}^*) \rangle$, for $i \in \{A, B\}$, where $\langle v_i(g_i, g_{-i}^*) \rangle$ denote the average payoff of i^{th} player when all the players, but i^{th} player, follow the strategy profile from g^* and i^{th} player follow some other strategy.

However, Aumann pointed out limitation about the achievability of Nash equilibrium: it can be achieved only when each of the players know other players' strategy exactly. He proposed a more general notion of equilibrium – correlated Nash equilibrium [13]. While in a mixed strategy players can choose pure strategies with probability $P(g_A, g_B) = P(g_A)P(g_B)$, with $P(g_i)$ denoting the probability distribution over the i^{th} player's pure strategy, Aumann pointed out that some adviser can provide a more general probability distribution (advice) which not necessarily is in the product form. A correlated strategy is defined as the map $\mathbf{g}(\lambda)$ chosen with some probability λ from the probability space Λ over $\mathcal{G} = \mathcal{G}_A \times \mathcal{G}_B$. The referee chooses an element λ from Λ and suggests to each player i to follow the strategy $g_i(\lambda)$. With such a advice from the

referee, the average payoff for the i^{th} player is denoted as $\langle v_i(g(\lambda)) \rangle \equiv \mathbb{E}_{t,g(\lambda)} v_i(t, g_A(t_A, \lambda), g_B(t_B, \lambda))$. A correlated strategy g^* chosen with some advice $\lambda \in \Lambda$ is called a correlated Nash equilibrium if no player has an incentive while deviating from the adopted strategy. Note that, every pure/mixed Nash equilibrium is also a correlated equilibrium, however the set correlation equilibria is strictly larger than the set of mixed strategy Nash equilibria [14]. It has also been shown that correlated equilibrium are easier to compute [31].

Quantum correlations as advice.– In quantum scenario the referee, instead of some classical correlation, provides some bi-partite quantum state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A^d \otimes \mathbb{C}_B^d)$ as advice; $\mathcal{D}(\mathbb{C}_A^d \otimes \mathbb{C}_B^d)$ denotes the set of density operator acting on the composite Hilbert space $\mathbb{C}_A^d \otimes \mathbb{C}_B^d$. The players perform positive-operator-valued-measurements (POVM) $\{E_{o_i}^{x_i} \mid E_{o_i}^{x_i} \geq 0 \forall o_i, x_i \sum_{o_i} E_{o_i}^{x_i} = \mathbb{1}_i \forall x_i\}$, with $\mathbb{1}_i$ being the identity operator on $\mathbb{C}_i^{d_i}$, and generate an input-output probability distribution $P(\mathcal{O}_A, \mathcal{O}_B | \mathcal{X}_A, \mathcal{X}_B) \equiv \{P(o_A, o_B | x_A, x_B) \mid o_i \in \mathcal{O}_i, x_i \in \mathcal{X}_i\}$, where the probabilities are achieved in accordance with the Born rule, i.e., $P(o_A, o_B | x_A, x_B) = \text{Tr}[\rho_{AB}(E_{o_B}^{x_B} \otimes E_{o_A}^{x_A})]$. The players follow some randomized strategy according to this probability distribution. In quantum scenario a strategy is specified by the triplet $(\rho_{AB}, \{E_{o_A}^{x_A}\}, \{E_{o_B}^{x_B}\})$. At this point it is important to note that, to have an advantage over the classical correlated strategies the correlation generated from a quantum strategy need to be stronger than classical (or in other word *local-realistic*) correlations Λ . From the study quantum foundations it is known that, if the given quantum advice ρ_{AB} is an entangled state [16, 17] then it can provide correlations which are not in the *local-realistic* form, and such correlation are commonly known as nonlocal correlation [18, 19]. In Bayesian game theoretic scenario usefulness of such nonlocal correlations over the classical correlated strategies has been shown in various recent results [20–22] and still remains an active area of research.

It is evident from the discussion in the previous paragraph that any quantum advice from the referee must be an entangled state in order to achieve better than the optimal classical equilibrium strategy. An entangled advice ρ_{AB}^{ent} will be called advantageous over a classical equilibrium strategy g^* if the players can come up with a quantum strategy $(\rho_{AB}^{\text{ent}}, \{E_{o_A}^{x_A}\}, \{E_{o_B}^{x_B}\})$ such that $\langle v_i(\rho_{AB}^{\text{ent}}) \rangle \geq \langle v_i(g^*) \rangle$, $\forall i$, and strict inequality hold for some (at least one) i . Here $\langle v_i(\rho_{AB}^{\text{ent}}) \rangle$ denotes the payoff for the i^{th} player while following the quantum strategy $(\rho_{AB}^{\text{ent}}, \{E_{o_A}^{x_A}\}, \{E_{o_B}^{x_B}\})$. Given a quantum advice ρ_{AB}^{ent} a strategy $(\rho_{AB}^{\text{ent}}, \{E_{o_A}^{x_A}\}^*, \{E_{o_B}^{x_B}\}^*)$ is optimal if no player has an incentive while deviating from the adopted strategy. A quantum advice $\rho_{AB}^{*\text{ent}}$ is called the optimal advice if there is a strategy $(\rho_{AB}^{*\text{ent}}, \{E_{o_A}^{x_A}\}^*, \{E_{o_B}^{x_B}\}^*)$ such

that no player has an incentive while deviating from the adopted strategy even with some other quantum advice. Such a strategy is called quantum equilibrium strategy. The authors in [21] have studied quantum equilibrium strategy in a conflicting Bayesian game. However the equilibrium studied there is a fair one where players have equal payoff. But, for the unfair case, where different players have different payoffs, though the notion of unfair classical equilibrium is well defined, but, as noted in [22], such a notion in quantum scenario is not pertinent, in general. This is because, given a quantum advice ρ_{AB}^{ent} , there may exist more than one quantum strategies, say $(\rho_{AB}^{ent}, \{E_{o_A}^{x_A}\}^{1^{st}}, \{E_{o_B}^{x_B}\}^{1^{st}})$ and $(\rho_{AB}^{ent}, \{E_{o_A}^{x_A}\}^{2^{nd}}, \{E_{o_B}^{x_B}\}^{2^{nd}})$, such that both are advantageous over the classical strategy g^* but Alice gets optimal payoff for 1^{st} strategy while Bob's payoff is optimal for 2^{nd} one and hence results to a conflict between the players in choosing their strategies for the given advice. In such a scenario, a relevant figure of merit for the unfair quantum strategies is social optimality solution or social welfare solution (SWS). The expected social welfare $SW(g)$ of a classical solution g is the sum of the expected payoffs of all the players, i.e., $SW(g) = \sum_i \langle v_i(g) \rangle$

[10]; this particular notion is also important in social choice theory [23]. Consider an classical unfair equilibrium solution g^* , with payoffs $\langle v_A(g^*) \rangle \neq \langle v_B(g^*) \rangle$. Among the different quantum advantageous strategies over g^* , a quantum strategy will be called quantum-SWS if it maximizes the sum of the payoffs. The corresponding quantum entangled state $\rho_{AB}^{ent-sws}$ producing the quantum-SWS is called quantum-SWA. To say mathematically, ρ_{AB}^{SWA} is a quantum-SWA if there exist some quantum strategy such that, $\langle v_i(\rho_{AB}^{SWA}) \rangle \geq \langle v_i(g^*) \rangle$, $\forall i$ (with strict inequality for some i), and the strategy maximize $\sum_i \langle v_i(\rho_{AB}^{SWA}) \rangle$. What we establish in the following, is that all the two-qubit pure entangled states are quantum-SWA for some Bayesian game.

Result.— Consider a game $G(\zeta, \eta)$ played between two rational players, Alice and Bob. Each of the players has two types, i.e., $t_i \in \mathcal{T}_i \equiv \{0, 1\}$ and two actions $s_i \in \mathcal{S}_i \equiv \{0, 1\}$, for $i \in \{A, B\}$. The players are to choose the types and actions from these sets. The payoffs assigned to the players depend on the respective types and actions: an utility table for the game $G(\zeta, \eta)$ is given in table-I. Since the utilities depend on the parameter ζ and η , we parametrize the game with these two parameters.

		$t_B = 0$		$t_B = 1$	
		$s_B = 0$	$s_B = 1$	$s_B = 0$	$s_B = 1$
$t_A = 0$	$s_A = 0$	$(\frac{\eta\zeta+1}{4}, \frac{\eta\zeta-1}{4})$	$(\frac{-2\eta+\eta\zeta+1}{4}, \frac{-2\eta+\eta\zeta-1}{4})$	$(\frac{2\eta+3}{4}, \frac{3\eta}{4})$	$(\frac{3}{4}, \frac{\eta}{4})$
	$s_A = 1$	$(0, 0)$	$(0, 0)$	$(\frac{3}{4}, \frac{\eta}{4})$	$(\frac{3}{4}, \frac{\eta}{4})$
$t_A = 1$	$s_A = 0$	$(\frac{-1}{4}, \frac{1}{4})$	$(0, 0)$	$(\frac{-\eta}{4}, \frac{-2\eta+9}{4})$	$(\frac{\eta}{4}, \frac{9}{4})$
	$s_A = 1$	$(\frac{-2\eta-1}{4}, \frac{-2\eta+1}{4})$	$(0, 0)$	$(\frac{\eta}{4}, \frac{9}{4})$	$(\frac{\eta}{4}, \frac{9}{4})$

Table I. (Color online) Utility table for the game $G(\zeta, \eta)$ with $\zeta \in [0, 2]$ and $\eta > 0$. Depending on the parameters ζ, η , the colored cells denotes different equilibria. When $1/(2-\zeta) < \eta < 1/\zeta$, there are two conflicting equilibrium strategies for the type $(t_A = 0, t_B = 0)$, that are $(s_A = 0, s_B = 0)$ and $(s_A = 0, s_B = 0)$ (blue cells). For $\eta > 1/2$ also, there are two conflicting strategies, i.e., $(s_A = 0, s_B = 0)$ and $(s_A = 0, s_B = 0)$ (yellow cells) for the type $(t_A = 1, t_B = 0)$.

From table-I one can see that following are the only possible pure Nash equilibrium strategies: (i) type $(t_A = 0, t_B = 0)$: in this case $(s_A = 0, s_B = 0)$ is an equilibrium strategy with payoff $(\eta\zeta+1)/4, (\eta\zeta-1)/4$, and whenever $\eta > 1/(2-\zeta)$ the strategy $(s_A = 1, s_B = 1)$ is also an equilibrium with payoff $(0, 0)$. Furthermore, if the values of the parameter ζ and η be such that $1/(2-\zeta) < \eta < 1/\zeta$, then there is conflict between Alice's and Bob's preferences: Alice prefers the strategy $(s_A = 0, s_B = 0)$ while Bob prefers $(s_A = 1, s_B = 1)$; (ii) type $(t_A = 0, t_B = 1)$: here $(s_A = 0, s_B = 0)$ and $(s_A = 1, s_B = 1)$ are two equilibria with payoffs

$((2\eta+3)/4, 3\eta/4)$ and $(3/4, \eta/4)$ respectively; (iii) type $(t_A = 1, t_B = 0)$: in this case $(s_A = 0, s_B = 0)$ is an equilibrium with payoff $(-1/4, 1/4)$, and whenever $\eta > 1/2$ there is another equilibrium, that is $(s_A = 1, s_B = 1)$ with payoff $(0, 0)$. Here also the equilibrium strategies are conflicting; (iv) type $(t_A = 1, t_B = 1)$: in this case there are three equilibria $(s_A = 0, s_B = 1)$, $(s_A = 1, s_B = 0)$ and $(s_A = 1, s_B = 1)$ with each of them having the payoff $(\eta/4, 9/4)$. Consider that types of the players are private, i.e., unknown to other player and hence the game is Bayesian in nature. Each player can choose the following four pure strategies:

$g_i^1(t_i) = 0$, $g_i^2(t_i) = 1$, $g_i^3(t_i) = t_i$, $g_i^4(t_i) = t_i \oplus 1$. Here $g_i^1(t_i) = 0$ means that i^{th} player follows the action $s_i = 0$ whatever her/his type t_i be, and other g_i 's are defined analogously where \oplus denotes addition modulo 2 operation. Therefore all together the players have 16 different pure strategies (g_A^l, g_B^m) , with $l, m = 1, 2, 3, 4$. Straightforward calculation gives the average payoffs for these 16 pure strategies and it turns out that classical equilibrium strategies have payoffs $\langle v_A(g^*) \rangle = (3 + \eta + \eta\zeta)/16$ and $\langle v_B(g^*) \rangle = (9 + \eta + \eta\zeta)/16$, respectively [14]

To establish our result, i.e., superlative behavior of all 2-qubit pure entangled states in the above described games, first we consider the set of most general 2-party-2-input-2-output no-signaling (NS) correlations that lies within a polytope, say \mathcal{P}_{NS} . Any such correlation $P(\mathcal{O}_A, \mathcal{O}_B | \mathcal{X}_A, \mathcal{X}_B) \equiv \{P(o_A, o_B | x_A, x_B)\} \in \mathcal{P}_{NS}$, with $o_i \in \mathcal{O}_i \equiv \{+1, -1\}$ and $x_i \in \mathcal{X}_i \equiv \{0, 1\}$ can be represented in a canonical form where $(P(++|00), P(+-|00), P(-+|00), P(--|00)) \equiv (c_{00}, m_0 - c_{00}, n_0 - c_{00}, 1 - m_0 - n_0 + c_{00})$ and the rests can be defined analogously [14]. When advised by such a correlation $P \in \mathcal{P}_{NS}$, Alice's and Bob's payoffs reads:

$$\langle v_i(P) \rangle = \frac{1}{16} \left[3^\kappa + \frac{\eta}{2} (\mathbb{B}_{CHSH} + 2\zeta m_0) - (-1)^\kappa (m_0 - n_0) \right], \quad (1)$$

with $\kappa = 1$ ($\kappa = 2$) for $i = A$ ($i = B$). Here, \mathbb{B}_{CHSH} denotes the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) expression, which for the given NS correlation P looks, $\mathbb{B}_{CHSH} := \sum_{k,j=0}^1 (-1)^{kj} \langle x_A = k, x_B = j \rangle = 4 \left(\sum_{k,j=0}^1 (-1)^{kj} c_{kj} - m_0 - n_0 + 1/2 \right)$, with $\langle x_A, x_B \rangle := \sum_{o_A, o_B=+1}^{-1} o_A o_B P(o_A, o_B | x_A, x_B)$. Correlations that are obtained from quantum strategies form a convex set, say \mathcal{Q} , which is a strict subset of the polytope \mathcal{P}_{NS} . As discussed earlier, a quantum strategy $(\rho_{AB}^{ent}, \{E_{o_A}^{x_A}\}, \{E_{o_B}^{x_B}\})$ will be a quantum social welfare solution for the game $\mathcal{G}(\zeta, \eta)$, if $\langle v_A(P) \rangle \geq \langle v_A(g^*) \rangle = (3 + \eta + \eta\zeta)/16$ and $\langle v_B(P) \rangle \geq \langle v_B(g^*) \rangle = (9 + \eta + \eta\zeta)/16$ (with at least one the inequalities strict) and $\langle v_A(P) \rangle + \langle v_B(P) \rangle$ takes the maximum value over the set of quantum correlations. Using the expression from Eq.(C1), we have,

$$\langle v_A(P) \rangle + \langle v_B(P) \rangle = \frac{1}{16} [12 + \eta (\mathbb{B}_{CHSH} + 2\zeta m_0)]. \quad (2)$$

Please note that, the factor within the round brackets on the right hand side of the Eq.(C1), i.e., the expression $\mathbb{B}_{CHSH} + 2\zeta m_0$, is actually the expression of tilted-CHSH operator studied in Ref.[24]. It has been shown in [25, 26] that within \mathcal{Q} the tilted-CHSH operator takes maximum value by a probability distribution $P(\mathcal{O}_A, \mathcal{O}_B | \mathcal{X}_A, \mathcal{X}_B) \in \mathcal{Q}$ obtained from the quantum state $|\psi\rangle_{AB} = \cos\theta|00\rangle_{AB} + \sin\theta|11\rangle_{AB}$ with the local projective measurement $E^{(x_A=0)} = \sigma_z$, $E^{(x_A=1)} = \sigma_x$ and

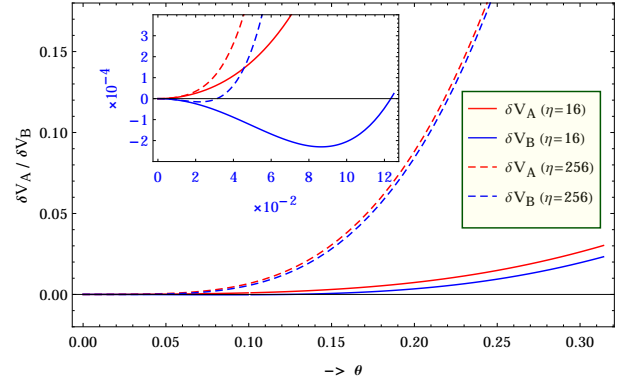


Figure 1. (Color online) δV_i vs θ plot. Solid curves are for $\eta = 16$, while dashed are for $\eta = 256$. Red for δV_A and blue for δV_B . For $\eta = 16$, δV_B is positive if θ is greater than ≈ 0.12 and for $\eta = 256$ it is positive if θ is greater than ≈ 0.03 (shown in the inset). δV_A is positive always.

$E^{(x_B=0)} = \cos\beta\sigma_z + \sin\beta\sigma_x$, $E^{(x_B=1)} = \cos\beta\sigma_z - \sin\beta\sigma_x$; where $\tan\beta = \sin 2\theta$ and $\zeta = 2/\sqrt{1 + 2\tan^2 2\theta} \in [0, 2]$. The same choice of state and measurements also maximize the right hand side of Eq.(C1). This is because, if $\mathbb{B} := \sum_{o_A, o_B, x_A, x_B} C_{o_A o_B x_A x_B} P(o_A, o_B | x_A, x_B) \leq \mathbb{B}_L$ is an arbitrary Bell operator with \mathbb{B}_L being the local bound, then the Bell operator $\mathcal{F}_{K_1, K_2}(\mathbb{B}) := K_1 \mathbb{B} + K_2$, with $K_1 \in \mathbb{R}_+$ and $K_2 \in \mathbb{R}$, has the local bound $\mathcal{F}_{K_1, K_2}(\mathbb{B}_L)$. Moreover the points on the boundary of set of quantum correlation that reaches the quantum maximum for \mathbb{B} and $\mathcal{F}_{K_1, K_2}(\mathbb{B})$ are going to be the same. This fact also ensures that for the games where i^{th} player's average payoff is of the form $\langle v_i(P) \rangle = \mathcal{F}_{K_1^i, K_2^i}(\mathbb{B})$, with some Bell operator \mathbb{B} but different K_j^i 's for different players', the concept of unfair equilibrium fits even in the quantum regime. However this is not the case always as happens for the game $\mathcal{G}(\zeta, \eta)$ considered in this work, and for this game the above mentioned optimal tilted-CHSH yields,

$$\langle v_i(P) \rangle = \frac{1}{16} \left[3^\kappa + \frac{\eta}{2} \frac{3 - \cos 4\theta}{\sqrt{1 + \sin^2 2\theta}} + \frac{2\eta \cos^2 \theta}{\sqrt{1 + 2\tan^2 2\theta}} - (-1)^\kappa \frac{1}{2} \cos 2\theta \left(1 - \frac{1}{\sqrt{1 + \sin^2 2\theta}} \right) \right]. \quad (3)$$

As already discussed a quantum strategy will be advantageous when the players have incentive over the classical equilibrium payoff, i.e., $\delta V_i := \langle v_i(P) \rangle - \langle v_i(g^*) \rangle \geq 0$ for $i \in \{A, B\}$, with strict inequality holding for at least one case. Taking the value of $\eta = 16$, we find that $\delta V_A > 0$ for the full range of the parameter $\theta \in (0, \pi/4]$, however δV_B remain positive if θ is not too small, if θ takes value greater than ≈ 0.12 (see Fig.1). Therefore the quantum states $|\psi_{AB}\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ corresponding to the

said range of θ act as the quantum social welfare advice for the game $\mathcal{G}(\zeta, \eta = 16)$, where $\zeta = 2/\sqrt{1+2\tan^2\theta}$. If we increase the value of η then δV_A remains always positive and δV_B becomes positive for even smaller values of θ (see Fig. 1). Moreover, taking arbitrarily large value for η one can make θ arbitrarily close to zero and can have quantum advantage [14]. It is also noteworthy that with increasing values for η the quantum advantage over classical payoff also increase. Therefore even when the given quantum entangled state is arbitrarily close to a product state still it suffices to be a quantum-SWA.

Discussions.— Study of entanglement, its quantification, classifications as well as its applications in different information theoretic protocols [27], is one of the core research topics of quantum information theory. Quantum entanglement also draws research attention from a foundational perspective since it lies at the core of some of the most puzzling features of quantum mechanics: the Einstein-Podolski-Rosen argument [28], the Schrödinger’s steering concept [29], and most importantly the nonlocal behavior of quantum mechanics [18, 19]. Here, we have studied application of this quantum information theoretic resource in another vastly important area of research field, Bayesian game theory. Our result establishes all two-qubit pure entangled states as the ‘gold coin’ in a certain Bayesian game theoretic scenario. From our analysis it is evident that the nonlocal behavior of the correlations obtained from those entangled states plays the key role in the

Bayesian scenario we have considered. This observation leads us to make some interesting comments based on some already known facts. In [30], the authors have shown that in the N -party-2-input-2-output scenario the quantum maximum of any linear Bell type expression, $\beta := \sum_{o_i, x_i, i \in \{1, \dots, N\}} C_{o_1, x_1, \dots, o_N, x_N} P(o_1, \dots, o_N | x_1, \dots, x_N)$, is achievable by measuring N -qubit pure states with projective observables. Therefore quantum strategies formed from these states and observables have the potential to be quantum-SWS for suitably chosen N -player Bayesian game where each player with two types and two action and where sum of the payoffs of the players turns out to be $\mathcal{F}_{K_1, K_2}(\beta)$. However, explicit construction of such games require extensive effort and promises to be an interesting topic for future research. Also note that the quantum-SWS studied in the $2-2-2$ scenario lie on the nonlocal boundary of the quantum set \mathcal{Q} . We leave the converse of the statement as a conjecture. We make the conjecture in a broader sense that any nonlocal boundary point of the set \mathcal{Q} for general $N-M-K$ scenario is a quantum-SWS for some Bayesian game.

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Appendix A: Nash equilibrium:- examples

To illustrate the idea of uncorrelated and correlated Nash equilibrium, here we discuss two examples.

Example-1: Our first example is the famous two-party game called 'battle of sexes' (BoS) where the payoffs of the players are given as in the table-II.

Table II. (Color Online) Utility table for the game of battle of sexes. Colored cells ($s_A = s_B$) are the two pure strategy Nash equilibria.

	$s_B = 0$	$s_B = 1$
$s_A = 0$	(2, 1)	(0, 0)
$s_A = 1$	(0, 0)	(1, 2)

The Nash equilibria are the action profile (same as strategy profile, since the players do not have multiple types) ($s_A = 0, s_B = 0$) with payoffs (2, 1) and the action profile ($s_A = 1, s_B = 1$) with payoffs (1, 2). Now in a practical scenario, Alice and Bob can follow an equilibrium strategy if each of them deterministically know the action of other party. But if the players have ignorance about others' strategy then the achievability of equilibrium strategies are in question. In such case, a referee can advice them to reach their goal. Let the referee tosses a coin and announces the outcome (head/tail) to both Alice and Bob. Upon receiving the outcome head (tail) each party follow the strategy

Table III. (Color Online) Utility table for the game of chicken. Colored cells ($s_A \neq s_B$) are the two pure strategy Nash equilibria.

	$s_B = 0$	$s_B = 1$
$s_A = 0$	(6,6)	(2,7)
$s_A = 1$	(7,2)	(0,0)

$s_i = 0$ ($s_i = 1$) and accordingly follow one of the equilibrium strategies. This example establishes clear practical usefulness of the idea of correlated equilibrium over the uncorrelated ones.

Example-2: To point out more drastic difference between uncorrelated and correlated Nash equilibrium, let us consider another game known as the ‘game of chickens’, specified by the payoff table-III. Here the Nash equilibria (uncorrelated) are $(s_A = 0, s_B = 1)$ with payoffs (2,7) and $(s_A = 1, s_B = 0)$ with payoffs (7,2). Also in this game there exists a uncorrelated mixed equilibrium strategy. If each player chooses the strategies $s_i = 0$ and $s_i = 1$ with probability $2/3$ and $1/3$, respectively then they have the equilibrium payoff $(14/3, 14/3)$. To see this, suppose player A (B) assigns probability p (q) to their respective pure action 0. The expected payoff for A (B) to $s_A = 0$ ($s_B = 0$) and $s_A = 1$ ($s_B = 1$) are respectively $4q + 2$ ($4p + 2$) and $7q$ ($7p$). From the definition of mixed strategy equilibrium it is evident that it will be attained when each will yield the same expected payoff for both $s_i = 0$ and $s_i = 1$ for $i = A, B$. This restricts both p and q to be $2/3$ to attain the expected payoff $(14/3, 14/3)$ for the mixed strategy equilibrium.

However like in the BoS game here also a referee can help the player to follow some particular correlated strategy. If the referee provides the players a correlation advice according to which they choose any one of pure strategies $(s_A = 0, s_B = 0)$, $(s_A = 0, s_B = 1)$, and $(s_A = 1, s_B = 0)$ randomly, then the average payoff will be (5,5) which is a correlated Nash equilibrium.

Note that, this correlated equilibrium can not be reached by convex mixing of the uncorrelated Nash equilibria. Clearly this shows that the notion of correlated equilibrium is more general than the original notion of equilibrium as introduced by Nash– correlated equilibrium can be in the outside of convex hull formed by the (uncorrelated) Nash equilibrium strategies. But it is important to point out that every Nash equilibrium is a correlated equilibrium though the converse is not true. Another fundamental aspect of game theory is the degree of complexity of finding the equilibria. It was shown that correlated equilibrium are easier to be computed [31].

Appendix B: Correlations (as Advice):- Local vs Nonlocal

Correlation obtained from the referee as advice helps the players to achieve the correlated equilibrium strategy. Based on different restrictions on the shared correlations, various notions of equilibrium can be defined, such as shared randomness equilibrium, no-signaling correlation equilibrium etc [12]. On the other hand, study of correlations, in particular local vs nonlocal as inspired by the seminal result of Bell [18], is one of the fundamental aspect of quantum foundations [19]. Very recently, Brunner and Linden have explored the connection between Bell nonlocality and Bayesian game theory [20]. In a Bayesian game each player may have some private information unknown to other players; on the other hand, the players may have a common piece of advice and thus can follow correlated strategies. As pointed out by Brunner and Linden, the concept of private information in Bayesian games is analogous to the notion of locality in Bell inequalities (BIs), and the fact that common advice in Bayesian games does not reveal the private information mimics the concept of no-signaling resources in case of BIs.

Correlations among spatially separated parties are relevant for our purpose. Any such correlations can be represented as input-output conditional probability distribution. Here, for our purpose, we restrict ourselves into two parties, Alice and Bob. Denoting the inputs of Alice and Bob by $x_A \in \mathcal{X}_A$ and $x_B \in \mathcal{X}_B$ and their outcomes by $o_A \in \mathcal{O}_A$ and $o_B \in \mathcal{O}_B$, the input-output probability can be represented as a conditional probability $P(\mathcal{O}_A, \mathcal{O}_B | \mathcal{X}_A, \mathcal{X}_B) := \{P(o_A o_B | x_A x_B) \mid o_A \in \mathcal{O}_A, o_B \in \mathcal{O}_B, x_A \in \mathcal{X}_A, x_B \in \mathcal{X}_B\}$ which must satisfy,

positivity: $P(o_A, o_B | x_A, x_B) \geq 0$, $\forall o_A, o_B, x_A, x_B$, and

normalization: $\sum_{o_A, o_B} P(o_A, o_B | x_A, x_B) = 1 \quad \forall x_A, x_B$.

Table IV. (Color Online) Average payoffs for 16 different pure strategies for the game $\mathbb{G}(\zeta, \eta)$.

	g_B^1	g_B^2	g_B^3	g_B^4
g_A^1	$\left(\frac{3+\eta+\eta\zeta}{16}, \frac{9+\eta+\eta\zeta}{16}\right)$	$\left(\frac{4-\eta+\eta\zeta}{16}, \frac{8-\eta+\eta\zeta}{16}\right)$	$\left(\frac{3+\eta+\eta\zeta}{16}, \frac{9+\eta+\eta\zeta}{16}\right)$	$\left(\frac{4-\eta+\eta\zeta}{16}, \frac{8-\eta+\eta\zeta}{16}\right)$
g_A^2	$\left(\frac{2-\eta}{16}, \frac{10-\eta}{16}\right)$	$\left(\frac{3+\eta}{16}, \frac{9+\eta}{16}\right)$	$\left(\frac{2-\eta}{16}, \frac{10-\eta}{16}\right)$	$\left(\frac{3+\eta}{16}, \frac{9+\eta}{16}\right)$
g_A^3	$\left(\frac{3+\eta+\eta\zeta}{16}, \frac{9+\eta+\eta\zeta}{16}\right)$	$\left(\frac{4-\eta+\eta\zeta}{16}, \frac{8-\eta+\eta\zeta}{16}\right)$	$\left(\frac{3-\eta+\eta\zeta}{16}, \frac{9-\eta+\eta\zeta}{16}\right)$	$\left(\frac{4+\eta+\eta\zeta}{16}, \frac{8+\eta+\eta\zeta}{16}\right)$
g_A^4	$\left(\frac{2-\eta}{16}, \frac{10-\eta}{16}\right)$	$\left(\frac{3+\eta}{16}, \frac{9+\eta}{16}\right)$	$\left(\frac{2+\eta}{16}, \frac{10+\eta}{16}\right)$	$\left(\frac{3-\eta}{16}, \frac{9-\eta}{16}\right)$

Correlations compatible with the principle of ‘relativistic causality’ principle or more generally ‘no signaling’(NS) principle which prevents instantaneous communication between two space-like separated locations need to satisfy further constraints:

$$P(o_B|x_A, x_B) := \sum_{o_A} P(o_A, o_B|x_A, x_B) = P(o_B|x_B), \quad \forall o_B, x_A, x_B; \quad (\text{B1a})$$

$$P(o_A|x_A, x_B) := \sum_{o_B} P(o_A, o_B|x_A, x_B) = P(o_A|x_A), \quad \forall o_A, x_A, x_B. \quad (\text{B1b})$$

Any such physical correlations obtained in classical world satisfy two further conditions called *locality* and *reality* and are of the following form [19]:

$$P(o_A, o_B|x_A, x_B) = \int \rho(\lambda) P(o_A|x_A, \lambda) P(o_B|x_B, \lambda) d\lambda, \quad (\text{B2})$$

where $\lambda \in \Lambda$ is some common shared variable sampled according to the probability distribution $\rho(\lambda)$. Correlations of the form of Eq.(B2) are also compatible with Reichenbach’s principle according to which if two physical variables are found to be statistically dependent, then there should be a causal explanation of this fact [32]. However, in 1966, in the seminal paper J.S. Bell came up with an inequality [18] which is satisfied by any local-realistic correlation of Eq.(B2). Interestingly, in his paper Bell also pointed out that in quantum world correlations can arise among the outcomes of measurements performed on the entangled states of space like separated particles that violate his inequality and such are called nonlocal.

1. 2-party–2-input–2-output NS correlations

Here we consider a more specific scenario with two inputs for each party with two outputs for each of the input, i.e., $o_i \in \mathcal{O}_i = \{0, 1\}$ and $x_i \in \mathcal{X}_i = \{0, 1\}$ for $i \in \{A, B\}$. We also consider that $\mathcal{T}_i = \mathcal{X}_i$ and $\mathcal{O}_i = \mathcal{S}_i$, that is i th player’s types and actions correspond, respectively, to the inputs and outputs of the NS correlation. The positivity and normalization constraints for 2-input 2-output scenario lead the probability vector to lie in a 8 dimensional polytope \mathcal{P}_{NS} [35]. Probability distributions satisfying the local-realistic constraint (B2) forms another polytope \mathcal{L} which is a strict subset of \mathcal{P}_{NS} . \mathcal{L} has both trivial and nontrivial facets– trivial facets correspond to the positivity constraints and the nontrivial ones to Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality [36]. The polytope \mathcal{P}_{NS} consists of 24 extremal points (vertices), where 16 of them are local deterministic points being the extremal points of \mathcal{L} and the rests 8 are nonlocal extremal points. The local boxes can be written as,

$$p^{\alpha, \beta, \gamma, \delta}(o_A, o_B|x_A, x_B) = \begin{cases} 1, & \text{if } o_A = \alpha x_A \oplus \beta \text{ and } o_B = \gamma x_B \oplus \delta \\ 0, & \text{otherwise,} \end{cases} \quad (\text{B3})$$

with $\alpha, \beta, \gamma, \delta \in \{0, 1\}$. The 16 pure strategies (g_A^l, g_B^m) , with $l, m = 1, 2, 3, 4$ described in the manuscript, actually correspond to these 16 local extremal points, i.e., the strategies are chosen according to these local deterministic extremal probability distributions.

The average payoffs for these 16 pure strategies are calculated in table-IV. There are three pure strategy Nash equilibria (g_A^1, g_B^1) , (g_A^3, g_B^1) , and (g_A^1, g_B^3) each average payoff $\langle v_A \rangle = (3 + \eta + \eta\zeta)/16$ for Alice and average payoff

$\langle v_B \rangle = (3 + \eta + \eta\zeta)/16$ for Bob. Since pure/mixed strategy Nash equilibrium are also correlated equilibrium hence these are also the correlated equilibria. Moreover any convex mixture of these equilibria are again a correlated equilibria but the average payoffs for both Alice and Bob takes the same values as in the pure cases.

The advice can also be the nonlocal extremal points given by,

$$P^{\alpha,\beta,\gamma}(o_A, o_B | x_A, x_B) = \begin{cases} 1/2, & \text{if } o_A \oplus o_B = x_A x_B \oplus \alpha x_A \oplus \beta x_B \oplus \gamma \\ 0, & \text{otherwise.} \end{cases} \quad (\text{B4})$$

with $\alpha, \beta, \gamma \in \{0, 1\}$, or more generally any correlation within \mathcal{P}_{NS} , that can be expressed as a 4×4 matrix in the following canonical form:

$$P(\mathcal{O}_A, \mathcal{O}_B | \mathcal{X}_A, \mathcal{X}_B) := \begin{pmatrix} c_{00} & m_0 - c_{00} & n_0 - c_{00} & 1 - m_0 - n_0 + c_{00} \\ c_{01} & m_0 - c_{01} & n_1 - c_{01} & 1 - m_0 - n_1 + c_{01} \\ c_{10} & m_1 - c_{10} & n_0 - c_{10} & 1 - m_1 - n_0 + c_{10} \\ c_{11} & m_1 - c_{11} & n_1 - c_{11} & 1 - m_1 - n_1 + c_{11} \end{pmatrix}, \quad (\text{B5})$$

where $(P(00|00), P(01|00), P(10|00), P(11|00)) \equiv (c_{00}, m_0 - c_{00}, n_0 - c_{00}, 1 - m_0 - n_0 + c_{00})$ and so on. Positivity constraint implies each element of the 4×4 matrix lies in between 0 and 1.

A correlation is known to be quantum one if it has a quantum realization, i.e., $P(o_A, o_B | x_A, x_B) = \text{Tr}[\rho_{AB}(E_{o_A}^{x_A} \otimes E_{o_B}^{x_B})]$, where $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A^d \otimes \mathbb{C}_B^d)$ and $\{E_{o_A}^{x_A}\}, \{E_{o_B}^{x_B}\}$ represents some local POVM on Alice's and Bob's side respectively. Collection of all quantum correlations \mathcal{Q} forms a convex set lying strictly in between \mathcal{P}_{NS} and \mathcal{L} , i.e., $\mathcal{L} \subset \mathcal{Q} \subset \mathcal{P}_{NS}$. Our main interest is to study social welfare solution within the set \mathcal{Q} for the the game $G(\zeta, \eta)$.

Appendix C: Pure entanglement as quantum-SWS:- calculation

If the two players are advised by a correlation from \mathcal{P}_{NS} the average payoff of each player turns out to be

$$\begin{aligned} \langle v_A(P) \rangle &= \frac{1}{16} \left[3 + \frac{\eta}{2} (\mathbb{B}_{CHSH} + 2\zeta m_0) + (m_0 - n_0) \right], \\ \langle v_B(P) \rangle &= \frac{1}{16} \left[9 + \frac{\eta}{2} (\mathbb{B}_{CHSH} + 2\zeta m_0) - m_0 + n_0 \right]. \end{aligned} \quad (\text{C1})$$

A quantum strategy will serve as a quantum social welfare solution if $\delta V_i := \langle v_i(P) \rangle - \langle v_i(g^*) \rangle \geq 0$ for $i = \{A, B\}$ and $\langle v_A(P) \rangle + \langle v_B(P) \rangle = \frac{1}{16} [12 + \eta (\mathbb{B}_{CHSH} + 2\zeta m_0)]$ yields the maximum value over \mathcal{Q} .

The maximum value within \mathcal{Q} of the term $\langle v_A(P) \rangle + \langle v_B(P) \rangle$ will be obtained when value of $(\mathbb{B}_{CHSH} + 2\zeta m_0)$ i.e. the tilted Bell-CHSH inequality, is maximum over \mathcal{Q} . The above expression will reach maximum for the quantum state $|\psi\rangle_{AB} = \cos\theta|00\rangle_{AB} + \sin\theta|11\rangle_{AB}$ with the local projective measurement $E^{(x_A=0)} = \sigma_z$, $E^{(x_A=1)} = \sigma_x$ and $E^{(x_B=0)} = \cos\beta\sigma_z + \sin\beta\sigma_x$, $E^{(x_B=1)} = \cos\beta\sigma_z - \sin\beta\sigma_x$; where $\tan\beta = \sin 2\theta$ and $\zeta = 2/\sqrt{1 + 2\tan^2 2\theta} \in [0, 2)$. As a result $m_0 = \cos^2\theta$, $n_0 = \frac{1}{2} \left(1 + \frac{\cos(2\theta)}{\sqrt{1 + \sin^2(2\theta)}} \right)$ and $\mathbb{B}_{CHSH} = (3 - \cos 4\theta)/\sqrt{1 + \sin^2 2\theta}$, which further imply,

$$\langle v_A(P) \rangle = \frac{1}{16} \left[3 + \frac{\eta}{2} \frac{3 - \cos 4\theta}{\sqrt{1 + \sin^2 2\theta}} + \frac{2\eta \cos^2 \theta}{\sqrt{1 + 2\tan^2 2\theta}} + \frac{1}{2} \cos 2\theta \left(1 - \frac{1}{\sqrt{1 + \sin^2 2\theta}} \right) \right], \quad (\text{C2a})$$

$$\langle v_B(P) \rangle = \frac{1}{16} \left[9 + \frac{\eta}{2} \frac{3 - \cos 4\theta}{\sqrt{1 + \sin^2 2\theta}} + \frac{2\eta \cos^2 \theta}{\sqrt{1 + 2\tan^2 2\theta}} - \frac{1}{2} \cos 2\theta \left(1 - \frac{1}{\sqrt{1 + \sin^2 2\theta}} \right) \right]. \quad (\text{C2b})$$

For the classical pure equilibrium strategies $g^* \equiv \{(g_A^1, g_B^1), (g_A^1, g_B^3), (g_A^3, g_B^1)\}$, the corresponding payoffs are,

$$\langle v_A(g^*) \rangle = \frac{3 + \eta + \eta\zeta}{16} = \frac{1}{16} \left(3 + \eta + \frac{2\eta}{\sqrt{1 + 2\tan^2 2\theta}} \right), \quad (\text{C3a})$$

$$\langle v_B(g^*) \rangle = \frac{9 + \eta + \eta\zeta}{16} = \frac{1}{16} \left(9 + \eta + \frac{2\eta}{\sqrt{1 + 2\tan^2 2\theta}} \right). \quad (\text{C3b})$$

For a given η , let θ_0 denotes the value of $\theta \in (0, \pi/4]$ beyond which δV_B takes positive value. In Fig.2 we show how the value of θ_0 tends towards zero with increasing values of η .

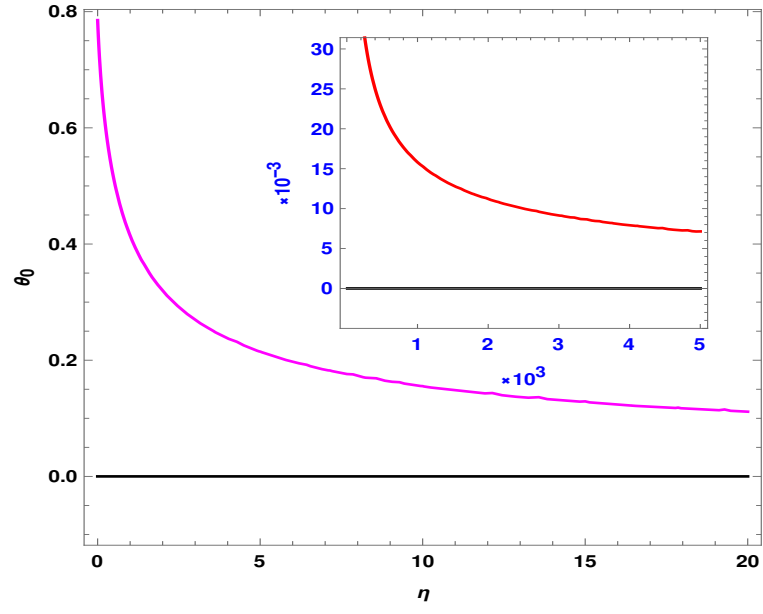


Figure 2. (Color online) θ_0 vs η plot. The graph shows that with increasing values of η the values of θ_0 gets decreased. The pink solid line is drawn for η taking values upto 20 and in the inset we plot it for η upto 5000.